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Real-Valued Characters and the Schur Index

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Let X be an irreducible complex character of the finite group G and let $m_R(X)$ denote the Schur index of X over the real numbers. It is known from the theory of the Schur index that $m_R(X) = 1$ or 2, and that $m_R(X) = 1$ if X is not real-valued. Let us now recall two basic facts concerning the real-valued irreducible complex characters of G. Firstly, the number of such characters equals the number of real classes of G, that is, conjugacy classes which contain the inverses of their elements. Secondly, for a real-valued complex irreducible character X, the number $\epsilon(X)$, defined by $\epsilon(X) = (1/|G|) \sum X(g^2)$, equals 1 if $m_R(X) = 1$ and -1 if $m_R(X) = 2$. This result is due to Frobenius and Schur, [3, p. 21], and it provides us with a simple procedure for determining the Schur index once the values of X are known. In [1, Problem 14], Brauer has asked for a description of the number of characters with $\epsilon(X) = 1$ purely in group theoretic terms. While an answer to Brauer's problem seems to be difficult, we will show that it is possible to give a lower bound for the number of characters with $\epsilon(X) = 1$ in terms of the number of real classes of odd order in the group (Theorem 1).

A contrary aspect of an investigation into the Schur index is to find sufficient conditions for the existence of real-valued irreducible characters with $\epsilon(X) = -1$. Our approach to this problem is based on the empirical fact that in many instances the existence of real-valued irreducible characters of Schur index 2 is accompanied by the existence of real elements which are not strongly real (an element of a group is said to be strongly real if it is inverted by an involution). However, we cannot assert that these two phenomena are equivalent, for it may happen that all elements of a group are strongly real while the group possesses real-valued characters of Schur index 2. An example is provided by the central product of quaternion and dihedral groups of order 8.

The main part of Section 2 is devoted to showing that, in certain circumstances, the existence of real elements of odd order which are not strongly real necessarily implies the existence of real-valued characters of Schur index 2. Our methods achieve their greatest effectiveness when the Sylow 2-subgroups of the group under investigation are abelian, and lead to an answer to Brauer's problem when the Sylow 2-subgroup is cyclic. We would emphasize that our analysis concentrates on the 2-regular structure of a group and is entirely ineffective when dealing with 2-groups. Indeed, 2-groups seem worthy of special attention and attempts to obtain a solution to Brauer's problem might reasonably begin with these groups. The paper ends with the observation that any 2-group which is not elementary abelian occurs as the Sylow 2-subgroup of a group with a real-valued character of Schur index 2.

1. Characters of Schur Index 1

Our intention in this section is to prove the following result.

THEOREM 1. *Let G be a finite group of even order which possesses t real classes of elements of odd order. Then G possesses at least $t + 1$ real-valued complex irreducible characters X with $\epsilon(X) = 1$.*

The author is indebted to the referee for showing how the author's original lower bound of t characters satisfying $\epsilon(X) = 1$ could be improved to $t + 1$ such characters.

Before embarking on a proof, we will introduce some notation. Let $|G| = 2^a b$ where $(b, 2) = 1$, and let Q_b be the field obtained by adjoining a primitive bth root of unity to the rational field Q. Let S be the ring of algebraic integers in Q_b and let P be a maximal ideal of S containing the prime 2.

Proof of Theorem 1. Our procedure consists initially of a modification of a lemma of Brauer, [2, p. 412]. Let $1 = h_1, \ldots, h_t$ be representatives of the t 2-regular real classes of G. We will show that there exist t real-valued complex irreducible characters $X_1, \ldots, X_t$, each of Schur index 1 over R, such that $\det X_i(h_j) \neq 0 \pmod{P}$, $1 \leqslant i, j \leqslant t$. Let us first note that G has at least one real-valued complex irreducible character of Schur index 1—the trivial character 1_G. If our statement above is not true, we can find elements $u_1, \ldots, u_t$ of S, with some $u_k \neq 0 \pmod{P}$, such that for any real-valued irreducible character X with $m_R(X) = 1$, $\sum_{i=1}^{t} u_i X(h_i) = 0 \pmod{P}$. Thus if θ is any S-linear combination of real-valued characters of Schur index 1 over R, we always have $\sum_{i=1}^{t} u_i \theta(h_i) = 0 \pmod{P}$.

We can take k to be greater than 1. For if this is not so, we have $u_j = 0 \pmod{P}$, $j \geqslant 2$, but $u_1 \neq 0 \pmod{P}$. Taking $X = 1_G$ in the sum $\sum u_i X(h_i)$, we obtain from the congruence of the first paragraph $u_1 = 0 \pmod{P}$, a contradiction. Thus we may take k to be greater than 1. Let H be the cyclic subgroup of G generated by $h = h_k$, with $|H| = 2n + 1$. H has $n + 1$ irreducible real representations with characters $\mu_1, \ldots, \mu_{n+1}$, say. We may describe these characters in the following way: μ_{n+1} is the trivial character, and if ω is a primitive $(2n + 1)$th root of unity, $\mu_i(h^j) = \omega^{ij} + \omega^{-ij}$. Let A denote the real character table of H. A is an $n + 1 \times n + 1$ matrix with entries $a_{ji} = \mu_i(h^j)$, $1 \leqslant i \leqslant n + 1$, $1 \leqslant j \leqslant n$, $a_{n+1,i} = \mu_i(1)$, $1 \leqslant i \leqslant n + 1$. Using the orthogonality relations for complex characters we may show that $(\det A)^2 = |H|^{n-1}$. It follows that there is a matrix B with entries in S, satisfying $AB = rI$, where r is in S and $r \not\equiv 0 \pmod{P}$.

We define the extended centralizer of h to be the subgroup $C^*(h) = \{x \in G: x\,h\,x^{-1} = h \text{ or } h^{-1}\}$. As h is a real element of G, $C^*(h)$ is an extension of $C(h)$ by a group of order 2. Let M^* be a Sylow 2-subgroup of $C^*(h)$ and let M be a subgroup of index 2 in M^* which centralizes h. We will show that the real characters $\mu_1, \ldots, \mu_{n+1}$ of H can be extended to $N = HM^*$. An extension of μ_{n+1} presents no problems. Extensions of the other characters may be obtained as follows. Take λ_i to be the complex character of H defined by $\lambda_i(h^j) = \omega^{ij}$, $1 \leqslant i \leqslant 2n$. Each λ_i may be extended to a character ν_i of HM by putting $\nu_i(h^j m) = \lambda_i(h^j)$, $m \in M$. The n induced characters $\sigma_i = \nu_i^N$, $1 \leqslant i \leqslant n$, are real-valued complex irreducible characters of N which extend the μ_i. It is easily shown, using the Frobenius-Schur method, for example, that the σ_i are realizable in the real field. Thus if $\rho_i = \sigma_i^G$, ρ_i is the character of a real representation of G.

Let ρ denote any one of the characters ρ_i. We may decompose ρ into a sum of complex irreducible characters of G,

$$\rho = \sum a_u X_u + \sum d_v X_v + \sum e_w X_w ,$$

where the first sum consists of characters which are not real-valued, the second of real-valued characters of Schur index 2, the third of real-valued characters of Schur index 1. Since ρ is real-valued, any nonreal constituent X of ρ occurs with the same multiplicity as its conjugate $\bar{X}$. Furthermore, each of the integers d_v must be even. For ρ is the character of a real representation of G and so $m_R(X_v)$ divides d_v, by a basic property of the Schur index. By definition of the characters in the second sum, $m_R(X_v) = 2$ and so each d_v is even. Thus we may write, with $d_v = 2f_v$

$$\rho = \sum a_u(X_u + \bar{X}_u) + \sum 2f_v X_v + \sum e_w X_w .$$

But as the elements $h_1, \ldots, h_t$ are real, $X_u(h_j) = \bar{X}_u(h_j)$, $1 \leqslant j \leqslant t$. We deduce that $\rho(h_j) \equiv \sum e_w X_w(h_j) \pmod{P}$, $1 \leqslant j \leqslant t$.

Let us put $\sigma = \sum_{i=1}^{n+1} b_i\rho_i$, where $b_1, \ldots, b_{n+1}$ are the entries of the first column of the matrix B introduced earlier. From the definition of induced characters, we have

$$\sigma(h_j) = (1/|N|) \sum_{x\in G} \sum_{i=1}^{n+1} b_i\sigma_i(xh_jx^{-1}),$$

where $\sigma_i(x\, h_j\, x^{-1})$ is set equal to 0 if $x\, h_j\, x^{-1}$ is not in N. If $x\, h_j\, x^{-1}$ belongs to N, it belongs to H, as it has odd order. Thus $\sigma_i(x\, h_j\, x^{-1}) = \mu_i(g)$ for some g in H. By the choice of $b_1, \ldots, b_{n+1}$, the value of the inner sum in the above formula is 0 unless $g = h$ or h^{-1}. It follows that $\sigma(h_j) = 0$ if $j \neq k$, $\sigma(h) = \sigma(h_k) = r \mid C^*(h): N \mid$. We note in particular that $\sigma(h) \not\equiv 0 \pmod P$.

By its construction, σ is an S-linear combination of complex characters of G. Let $\theta_1, \ldots, \theta_q$ be the real-valued irreducible constituents of σ which are of Schur index 1 over R. The properties of the ρ_i already deduced show that $\sigma(h_j) = \sum_{i=1}^{q} g_i\theta_i(h_j) \pmod P$, $1 \leqslant j \leqslant t$, where the $g_i \in S$. Thus if $\theta = \sum g_i\theta_i$, θ is an S-linear combination of real-valued irreducible characters of Schur index 1 over R with $\theta(h_j) = 0 \pmod P$, $j \neq k$, $\theta(h_k) \neq 0 \pmod P$. But by definition of the u_i of the opening paragraph, we must have $\sum_{i=1}^{t} u_i\theta(h_i) = 0 \pmod P$, which leads to the conclusion that $u_k\theta(h_k) = 0 \pmod P$. This is impossible as neither u_k nor $\theta(h_k)$ is zero modulo P and the existence of t characters $X_1, \ldots, X_t$ with $\epsilon(X_i) = 1$, $1 \leqslant i \leqslant t$, and $\det X_i(h_j) \neq 0 \pmod P$ is established.

Finally, we have to show that in fact there are at least $t + 1$ real-valued irreducible characters X with $\epsilon(X) = 1$. Let us suppose, by way of contradiction, that there are exactly t such characters. These must be the t characters $X_1, \ldots, X_t$ already shown to exist. We will take $X_1 = 1_G$. Let T be a Sylow 2-subgroup of G and let λ be a nontrivial linear character of T with $\lambda^2 = 1_T$. Let $\phi = \lambda^G$ and $\psi = 1_T{}^G$. Both ϕ and ψ are characters of real representations of G. Thus if $\phi = \sum_{i=1}^{t} \alpha_iX_i + \sum_{j>t} \alpha_jX_j$ gives the decomposition of ϕ into irreducible characters of G, and if we define ϕ_0 to equal $\sum_{i=1}^{t} \alpha_iX_i$, previous arguments give $\phi = \phi_0 \pmod P$ on $h_1, \ldots, h_t$. If we similarly define $\psi_0 = \sum_{i=1}^{t} \beta_iX_i$, we obtain $\psi = \psi_0 \pmod P$ on $h_1, \ldots, h_t$. However, $\phi(h_i) = \psi(h_i) = 0$ for $i \geqslant 2$ and $\phi(1) = \psi(1)$. Thus $\phi - \psi = \phi_0 - \psi_0 = 0 \pmod P$ on $h_1, \ldots, h_t$. It follows that $\sum (\alpha_i - \beta_i)\, X_i(h_j) = 0 \pmod P$, $1 \leqslant i, j \leqslant t$. As $\det X_i(h_j) \neq 0 \pmod P$, we obtain $\alpha_i = \beta_i \pmod P$. But as $X_1 = 1_G$, it follows from Frobenius reciprocity that $\alpha_1 = 0$, $\beta_1 = 1$ and thus $\alpha_1 - \beta_1 = -1 \neq 0 \pmod P$, a contradiction. Thus there are at least $t + 1$ characters with $\epsilon(X) = 1$ and our theorem follows.

Without additional hypotheses on the group G, our lower bound of Theorem 1 is best possible. For if we consider the group $G = SL(2, 2^n)$, we find that G has $2^n + 1$ conjugacy classes, all of which are real. Moreover

2^n of the classes are 2-regular. Thus our theorem predicts that all characters X of G satisfy $\epsilon(X) = 1$, a fact which is borne out by applying the Frobenius-Schur formula.

Note. Theorem 1 is of some interest for the theory of modular representations. For, using Brauer's permutation lemma, it may be proved that if the group G possesses t real 2-regular classes, it has exactly t absolutely irreducible 2-modular real-valued Brauer characters. Let $\phi_1, \ldots, \phi_t$ be these Brauer characters and let $\phi_{t+1}, \ldots, \phi_r$ be the remaining irreducible Brauer characters. Let $X_1, \ldots, X_t$ be characters of G which satisfy the hypotheses of the first paragraph of the proof. Then we can write $X_i = \sum_{u=1}^{t} d_{iu}\phi_u + \sum_{v=t+1}^{r} d_{iv}\phi_v$ on 2-regular elements of G, where the d_{ij} are the decomposition numbers of the X_i. Since X_i is real-valued, a Brauer character ϕ_v which is not real-valued must occur with the same multiplicity as its conjugate $\bar{\phi}_v$ in the decomposition of X_i. Following previous arguments, we may proceed to deduce that $X_i(h_j) \equiv \sum_{u=1}^{t} d_{iu}\phi_u(h_j) \pmod{P}$, $1 \leqslant i, j \leqslant t$. Thus we must have det $\phi_i(h_j) \not\equiv 0 \pmod{P}$, $1 \leqslant i, j \leqslant t$, which implies that the real-valued Brauer characters are distinguished by the values they take on real 2-regular elements.

2. Characters of Schur Index 2

Our objective in this section is the proof of the following result.

THEOREM 2. *Let G be a finite group which possesses r 2-regular real classes which are not strongly real. Let $h_1, \ldots, h_r$ be representatives of these classes. Then if $C(h_i)$ has an abelian Sylow 2-subgroup, $1 \leqslant i \leqslant r$, G has r real-valued complex irreducible characters $X_1, \ldots, X_r$ of Schur index 2 over R such that* det $X_i(h_j) \not\equiv 0 \pmod{P}$, $1 \leqslant i, j \leqslant r$.

The proof of this theorem is similar to the proof of Theorem 1, but requires some preliminary work. We begin by defining a certain metacyclic group which will play an important role in the subsequent development of our ideas. For each odd positive integer $t \geqslant 3$, we define B_{4t} to be the group $\{a, b : a^t = 1, b^4 = 1, a^b = a^{-1}\}$. $B = B_{4t}$ has a center Z of order 2 generated by b^2 and B/Z is dihedral of order $2t$. B has 4 linear characters and $t - 1$ characters of degree 2. Since B has only one involution, whereas B/Z has t involutions, a simple application of the Frobenius-Schur involution formula, [3, p. 22], shows that all $t - 1/2$ characters of B which are of degree 2 and faithful for Z are real-valued but of Schur index 2 over R. We may describe these characters in the following way: Let A be the normal subgroup of B generated by a and let λ be a nontrivial linear character of A. We may extend

λ to a character μ of AZ by putting $\mu(cb^2) = -\lambda(c)$ for $c \in A$. The induced character μ^B is a real-valued character of B of Schur index 2.

The following lemma is a simple consequence of the Frobenius–Schur involution formula, and we omit its proof.

LEMMA 1. *Let G be a finite group which possesses a normal subgroup H. If G/H has more involutions than G, G has real-valued irreducible characters of Schur index 2 over R.*

With this result, we can proceed to a proof of a technical lemma.

LEMMA 2. *Let G be a finite 2-group which possesses an abelian subgroup H of index 2. Suppose that G does not split over H. Then either G has a normal subgroup $K < H$ with G/K cyclic of order 4 or a normal subgroup $L < H$ with G/L generalized quaternion.*

Proof. Let us suppose that G does not have a homomorphic image which is generalized quaternion. It follows that no homomorphic image of G has more involutions than G. For if this is not so, Lemma 1 implies that G has a real-valued irreducible character of Schur index 2. This character must be of degree 2, since G has an abelian subgroup of index 2. Therefore, G has a homomorphic image G/L which is a subgroup of the symplectic group $Sp(2, C)$. However, it is easily shown that a finite irreducible 2-subgroup of $Sp(2, C)$ is generalized quaternion, which contradicts our previous supposition.

We note that as G does split over H, G has the same number of involutions as H, and H being abelian, this number is the order of the Frattini factor group of H. Thus if M is the Frattini subgroup of H, G/M has the same number of involutions as G and H. Our arguments will apply to G/M, so we may as well assume that M is 1 and H is elementary abelian. Let y be an element of $G - H$. We know from the theory of linear transformations that a basis $x_1, \ldots, x_t$ of H may be chosen so that $y^{-1}x_{2i-1}y = x_{2i-1}x_{2i}$, $y^{-1}x_{2i}y = x_{2i}$, $1 \leqslant i \leqslant S$, $y^{-1}x_jy = x_j$, $2S + 1 \leqslant j \leqslant t$. Let $y^2 = x_1^{a_1} \cdots x_t^{a_t}$. Since G does not split over H, there must be an index $j \geqslant 2S + 1$ with $a_j = 1$. For, writing an element $x_1^{a_1} \cdots x_{2S}^{a_{2S}}$ in the form $(a_1, \ldots, a_{2S})$, if $y^2 = (a_1, \ldots, a_{2S})$, we find that $x = y(a_2, 0, \ldots, a_{2S}, 0)$ is an involution of G which is not in H. To see this, we note that $x^2 = y^2(a_2, a_2, \ldots, a_{2S}, a_{2S})$ $(a_2, 0, \ldots, a_{2S}, 0) = (a_1, 0, \ldots, a_{2S-1}, 0)$. But as y must commute with y^2, we must have $a_1 = \cdots = a_{2S-1} = 0$. Thus x is indeed an involution, which we know cannot be the case. This contradiction shows that there is some $j \geqslant 2S + 1$ with $a_j = 1$. Let K be subgroup of H generated by all the x_i except x_j. K is normal in G, since y centralizes x_j. Finally, we note that

G/K is cyclic of order 4, for $y^2K = x_jK \neq K$. Thus we have proved the existence of homomorphic images of the desired type.

Proof of Theorem 2. Let us first show that G has at least one real-valued irreducible character of Schur index 2. Let h be a real 2-regular element of G which is not strongly real such that $C(h)$ has an abelian Sylow 2-subgroup. Since h is not strongly real, $C^*(h)$ is a nonsplit extension of $C(h)$. Thus if T is a Sylow 2-subgroup of $C^*(h)$ and $W = T \cap C(h)$, T is a nonsplit extension of W. Let H be the cyclic subgroup of G generated by h, with $|H| = 2n + 1$, and let $N = HT$.

Let $\mu_1, \ldots, \mu_n$ be the characters of the n nontrivial irreducible real representations of H. In Theorem 1 we showed that these characters can be extended to characters of real representations of N. The proof of Theorem 2 requires us to show that the μ_i can also be extended to real-valued irreducible characters of N which are of Schur index 2. To this end, let us first suppose that T has a normal subgroup $V < W$ with T/V cyclic of order 4. V is normal in N, for H must centralize V. Since each element of $T - W$ inverts h, we see that N/V is isomorphic to the group $B_{4(2n+1)}$ introduced earlier. The n characters of degree 2 which are nontrivial on W/V are irreducible real-valued characters of N of Schur index 2 which extend the μ_i.

If T has no such subgroups V, it follows from Lemma 2 that T has a normal subgroup $U < W$ with T/U generalized quaternion. We will construct n characters of N/U with the desired extension property, and, for the sake of brevity, we may suppose that $U = 1$. Thus W is a normal cyclic subgroup of T and HW is a normal cyclic subgroup of N. The automorphism of W induced by T is that which sends each element to its inverse. Since each element of H is also inverted by T, we see that the automorphism of HW induced by N is also $x \to x^{-1}$. Now let $\lambda_1, \ldots, \lambda_n$ and their inverses be the nontrivial complex characters of H and let μ be any faithful linear character of W. Then if $\nu_i = \lambda_i\mu$, $1 \leqslant i \leqslant n$, ν_i is a character of HW and the induced character $\nu_i^N = \sigma_i$ of N is real-valued and irreducible. Evidently the σ_i are extensions of the μ_i, and since $(\sigma_i)_T = \mu^T$ is a faithful irreducible character of T, and thus of Schur index 2 over R, σ_i is itself of Schur index 2 over R. We have obtained the desired extension of the μ_i.

The preceding two paragraphs have shown that, whatever the precise structure of T may be, there are n real-valued irreducible characters, $\sigma_1, \ldots, \sigma_n$, of N, of Schur index 2, which extend the characters $\mu_1, \ldots, \mu_n$ of H. Let $\theta = \theta_i$ be the character of G induced by $\sigma = \sigma_i$. As σ is real-valued, so is θ. Thus if X is a nonreal irreducible constituent of θ, its conjugate $\bar{X}$ occurs in θ with the same multiplicity. Let X be a real-valued irreducible constituent of θ. If $m_R(X) = 1$, X must occur with even multiplicity. For, suppose that this is not so and that X occurs with odd multiplicity b. Then by reciprocity σ occurs b times in X_N. As X is realizable in the real field, a basic property of the Schur index [3, p. 62], shows that $m_R(\sigma)$ divides b. But we already know that $m_R(\sigma) = 2$ and so b cannot be odd. Thus using the notation of our previous theorem we have $\theta(h) = 0 \pmod{P}$ if θ contains no irreducible characters of Schur index 2. For if θ contains no such characters, our deductions above show that we can write $\theta = \sum a_j(X_j + \bar{X}_j) + \sum 2b_k X_k$, where the first sum consists of nonreal-valued characters, and the second of real-valued characters of index 1. Since h is real, $X_j(h) = \bar{X}_j(h)$ and thus $\theta(h) = 2\sum a_j X_j(h) + 2\sum b_k X_k(h) = 0 \pmod{P}$.

The next step of the proof is to show that we cannot have $\theta_i(h) = 0 \pmod{P}$ for all n characters θ_i. This may be done by using the methods of our previous theorem. Let D be the $n \times n$ matrix with entries $d_{ij} = \mu_j(h^i)$, $1 \leqslant i, j \leqslant n$. D may be recognized as an $n \times n$ submatrix of the real character table of H. It may be established that $(\det D)^2 = |H|^{n-1}$ and so there is a matrix E with entries in S such that $DE = \lambda I$, where $\lambda \in S$, $\lambda \neq 0 \pmod{P}$. We set $\rho = \sum_{i=1}^{n} e_i \theta_i$, where $e_1, \ldots, e_n$ are the entries of the first column of E. Our previous proof may be invoked to show that $\rho(h) = \lambda \mid C^*(h): N \mid$. If each $\theta_i(h) = 0 \pmod{P}$, $1 \leqslant i \leqslant n$, it would follow that $\rho(h) = 0 \pmod{P}$, which is not true. Thus some $\theta_i(h) \neq 0 \pmod{P}$ and so G has a real-valued irreducible character of Schur index 2.

We are in a position to finish the proof. Corresponding to each element h_i, $1 \leqslant i \leqslant r$, we construct a generalized character ρ_i as above, satisfying $\rho_i(h_j) = 0 \pmod{P}$, $i \neq j$, $\rho_i(h_i) \neq 0 \pmod{P}$. Let $X_1, \ldots, X_u$ be the real-valued irreducible characters of G of Schur index 2. Then if ρ_i contains the character X_k with multiplicity b_{ik}, where $b_{ik} \in S$, our previous work gives

$$\rho_i(h_j) = \sum_{k=1}^{u} b_{ik} X_k(h_j) \pmod{P}.$$

Since the $r \times r$ matrix $\rho_i(h_j)$, $1 \leqslant i, j \leqslant r$, is invertible $\pmod{P}$, the $u \times r$ matrix $X_k(h_j)$, $1 \leqslant k \leqslant u$, $1 \leqslant j \leqslant r$, must have rank $r \pmod{P}$. This implies that $u \geqslant r$ and, additionally, with a suitable renumbering of the X_i, if necessary, that $\det X_i(h_j) \neq 0 \pmod{P}$, $1 \leqslant i, j \leqslant r$. This completes the proof.

It would be of interest to know to what extent the hypotheses of Theorem 2 may be weakened without altering the conclusions concerning the existence of characters of Schur index 2. However, we would point out the following example. There is group G of order 48, with a normal Sylow 3-subgroup generated by an element h. $C(h)$ has a quaternion Sylow 2-subgroup of order 8 and $C^*(h) = G$ has a quaternion subgroup of order 16. We find that G has 3 irreducible real-valued characters of Schur index 2 but that they all satisfy $X(h) = 0 \pmod P$. Thus the precise statement of Theorem 2 cannot be proved be proved without some assumption concerning centralizers.

There certainly do exist groups which satisfy the hypotheses of our theorem. For example, if q is a power of an odd prime p, the simple group $G = PSU(3, q^2)$ has an element h of order p which is real but not strongly real. $C^*(h)$ is of order $2q^3(q+1)$ and has a cyclic Sylow 2-subgroup. Thus G has at least one real-valued character of Schur index 2. We find that G has only one such character and its degree is $q(q-1)$. Finally, we note that if q is odd, all 2-regular real elements of $SL(2, q)$ satisfy the hypotheses of Theorem 2.

3. Groups with Abelian Sylow 2-subgroups

Theorem 2 is always applicable if a group possesses an abelian Sylow 2-subgroup. We will show in this section that, for a group with an abelian Sylow 2-subgroup, the existence of real-valued characters of index 2 and the existence of real 2-regular elements which are not strongly real are related phenomena. Our immediate aim is the proof of the following result.

THEOREM 3. *Let G be a finite group whose Sylow 2-subgroup is abelian. Then if G possesses real-valued irreducible characters of Schur index 2 over R, G possesses real 2-regular elements which are not strongly real.*

For the purpose of the proof, let us recall that a group H is said to be R-elementary at the prime 2 if:

(1) H is a semidirect product AB of a 2-group B and a cyclic $2'$-group $A = [h]$.

(2) For each $b \in B$, $bhb^{-1} = h$ or h^{-1}.

Proof of Theorem 3. Let X be an irreducible real-valued character of G of Schur index 2 over R. By [3, 15.12, p. 84], there is a subgroup H of G which is R-elementary at the prime 2 and an irreducible real-valued character θ of H such that (θ, X_H) is odd. The character θ must itself be of Schur index 2 over R. Writing $H = AB$ as above, B is abelian as the Sylow 2-subgroup of G is abelian. H cannot be a direct product of A and B, for it would follow that H is abelian and thus θ, being automatically linear, would be of Schur index 1. Thus there is some $b \in B$ with $bhb^{-1} = h^{-1}$. We will show that h cannot be strongly real.

Suppose, by way of contradiction, that h is strongly real, with $tht^{-1} = h^{-1}$ for some involution t. Since H is contained in $C^*(h)$, we can find a Sylow 2-subgroup T of $C^*(h)$ containing B. Let W be a subgroup of index 2 in T

which centralizes h. T splits over W, since $C^*(h)$ splits over $C(h)$ (t is an involution of $C^*(h)$ not in $C(h)$). Let $N = AT$, a subgroup containing H. N has a normal abelian subgroup $M = AW$ of index 2, and, since T splits over W, N splits over M. It now follows from [3, proof of 11.7, p. 64] that all characters of N have Schur index 1 and degree at most 2.

Finally, we consider the decomposition of X_N into irreducible characters of N. From previous arguments, $X_N = \sum a_j(\lambda_j + \bar{\lambda}_j) + \sum b_j\mu_j$, where the first sum consists of characters which are not real-valued, the second of real-valued characters. Since all real-valued characters of N have Schur index 1, we know that each b_j must be even (otherwise we would obtain $m_R(X) = 1$). Let us note that any irreducible character of N whose restriction to H contains θ must be an extension of θ. Moreover, if $\lambda_j = \theta$ on H, $\bar{\lambda}_j = \theta$ on H also. Thus we see that $X_H = (X_N)_H$ must contain θ an even number of times, which is not the case. Our conclusion is that h cannot be strongly real, as required.

Burnside's conjugacy criterion [4, 2.5, p. 418], shows that in a group G whose Sylow 2-subgroup is abelian, the only real 2-elements of G are involutions. From this we readily deduce that an element of G is strongly real if and only if its 2-regular component is strongly real. Thus, combining Theorems 2 and 3, we obtain:

COROLLARY 1. *Let G be a finite group whose Sylow 2-subgroup is abelian. Then all real-valued irreducible characters of G have Schur index 1 over R if and only if all real elements of G are strongly real.*

For a group with a cyclic Sylow 2-subgroup, it is possible to solve Brauer's problem, as we will show.

LEMMA 3. *Let G be a finite group with a cyclic Sylow 2-subgroup of order at least 2. Suppose that G has r real classes of elements of odd order, t of which consist of elements which are not strongly real. Then G has $r + t + 1$ real classes.*

Proof. Let g be a real element of G of even order. We can write g uniquely in the form $g = bc = cb$, where b is 2-regular, c is a 2-element. Since c must be real, we know that c has to be an involution. All involutions of G are conjugate, so we have to show that G has t real classes of elements of even order greater than 2. Thus we will assume that the 2-regular component b of g is nontrivial and will proceed to show that b is not strongly real.

As g is real, there is an element h in G with $g^h = g^{-1}$, and thus $b^hc^h = b^{-1}c$. By the uniqueness of the decomposition of g, $b^h = b^{-1}$, and it follows that b is real element, centralized by the involution c. In particular, $| C^*(b)$:

$C(b)| = 2$. The Sylow 2-subgroup of $C^*(b)$ is of course cyclic and so all involutions of $C^*(b)$ are conjugate. However, the involution c is $C(b)$ and consequently all involutions of $C^*(b)$ are in $C(b)$. We deduce that b is not strongly real and that all real elements of even order greater than 2 are products of involutions and real 2-regular elements which are not strongly real. Conversely, we may easily show that if b is a real 2-regular element which is not strongly real, there is an involution c in $C(b)$ with bc a real element of even order greater than 2.

There remains the problem of finding the number of classes of real elements of even order. Let b_1c_1, b_2c_2 be two real elements of even order greater than 2, where the b_i are 2-regular, the c_i involutions and $b_ic_i = c_ib_i$, $1 \leq i \leq 2$. Evidently, if b_1c_1 is conjugate to b_2c_2, b_1 is conjugate to b_2. Conversely, if b_1 is conjugate to b_2, there is an element x of G with $x^{-1}b_1x = b_2$ and $x^{-1}C(b_1)x = C(b_2)$. We note that $x^{-1}c_1x$ and c_2 are involutions of $C(b_2)$ and hence are conjugate in $C(b_2)$. We then have $x^{-1}c_1x = y^{-1}c_2y$ for some y in $C(b_2)$. Thus $yx^{-1}b_1c_1xy^{-1} = b_2c_2$ and we see that b_1c_1 is conjugate to b_2c_2. We have shown that there are t classes of real elements of even order greater than 2 and our lemma is established.

We may calculate the precise number of real-valued characters with $\epsilon(X) = 1$ in a group with a cyclic Sylow 2-subgroup using Theorems 1 and 2 and Lemma 3.

THEOREM 4. *Let G be a finite group with a cyclic Sylow 2-subgroup of order at least 2. Then if G has exactly r real 2-regular classes, G has exactly $r + 1$ real-valued irreducible characters with $\epsilon(X) = 1$.*

Proof. By Theorem 1, G has at least $r + 1$ irreducible characters with $\epsilon(X) = 1$. By Theorem 2, we know that if G has t real 2-regular classes which are not strongly real, G has at least t real-valued irreducible characters of Schur index 2. Thus G has at least $r + t + 1$ real-valued irreducible characters. However, Lemma 3 states that G has $r + t + 1$ real classes, and hence exactly $r + t + 1$ real-valued irreducible characters. Therefore G has exactly $r + 1$ real-valued characters with $\epsilon(X) = 1$.

4. ARBITRARY SYLOW 2-SUBGROUPS

We note that if a group G has an elementary abelian Sylow 2-subgroup, all real elements of G must be strongly real. Thus it follows from Theorem 3 that in such a group all real-valued irreducible characters have Schur index 1 over R. It is reasonable to ask whether there are any other classes of 2-groups for which such a statement may be made. However, we may answer this in the negative by showing:

PROPOSITION. *Let T be a 2-group which is not elementary abelian. Then there exists a group G, with Sylow 2-subgroup isomorphic to T, which possesses a real-valued irreducible character of Schur index 2 over R.*

Proof. We will construct a group G which is a split extension of an elementary abelian p-group by a 2-group isomorphic to T, p being any odd prime. Let T be a counter-example of minimal order to the statement of our proposition. We will first show that in this case all proper factor-groups of T are elementary abelian. For if U is a factor group of T which is not elementary abelian, the minimality of T forces the existence of an elementary abelian p-group N and an action of U on N such that $H = NU$ has an irreducible character of Schur index 2 over R. If $T/K = U$, we can define an action of T on N, wherein K centralizes N and T/K acts on N as U does. We now see that $G = NT$ has a factor-group isomorphic to H and so G itself has a real-valued irreducible character of Schur index 2 over R. As this contradicts the way in which T was defined, all proper factor-groups of T are elementary abelian.

Let x be an element of order 2 in the center of T, and let Z be the subgroup of T generated by x. T is not elementary abelian, so there exists an element y of order 4 in T, and since T/Z has exponent 2, $y^2 \in Z$. Thus we must have $y^2 = x$. Let $x_1Z,\ldots, x_nZ$ be generators of T/Z with $x_1 \cdots x_nZ = yZ$. We will define an action of T on an elementary abelian p-group N of order p^n in the following manner: Take generators $c_1, \ldots, c_n$ of N and set $c_i^x = c_i$, all i, $c_i^{x_i} = c_i^{-1}$, $c_i^{x_j} = c_i$, $i \neq j$. In this action, Z centralizes N and T/Z acts faithfully. We also note that $c^y = c^{-1}$ for any element c of N.

The last step of the proof is the construction of a character of $G = NT$ which satisfies the statement of the proposition. We first define a character λ of N by putting $\lambda(c_1) = \cdots = \lambda(c_n) = w$, where w is a primitive pth root of unity. It is easy to check that the stabilizer of λ in G is the subgroup NZ. We extend λ to a character μ of NZ by putting $\mu(xc) = -\lambda(c)$, $c \in N$. As λ has 2^n conjugates in G, $\mu^G = X$ is irreducible of degree 2^n, and is easily seen to be real-valued. We will show that $m_R(X) = 2$. To this end, we let Y be the subgroup generated by y and put $M = NY$. Since y inverts all elements of N, $\theta = \mu^M$ is irreducible and real-valued. Moreover as $\theta^G = X$, $m_R(X) = m_R(\theta)$. But we may show that θ is essentially a faithful character of the group B_{4p} which we introduced in Section 2. We know that θ must therefore satisfy $m_R(\theta) = 2$ and so $m_R(X) = 2$. Thus no such minimal counter-example exists and the theorem is true.

In conclusion, we suggest that it may be possible to extend Corollary 1 to some other groups whose Sylow 2-subgroups are not too complicated. For example, we may show that if G has a dihedral Sylow 2-subgroup, then all real-valued irreducible characters of G satisfy $\epsilon(X) = 1$ if and only if each real 2-regular class of G is strongly real. We also note that the Brauer-Suzuki theorem may be used to show that a group whose Sylow 2-subgroup is generalized quaternion always has an irreducible character with $\epsilon(X) = -1$.

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